One-step Methods

Numerical Solutions for ODEs

One-step Methods

Solving ODEs of the form: $\frac{dy}{dx} = f(x, y)$

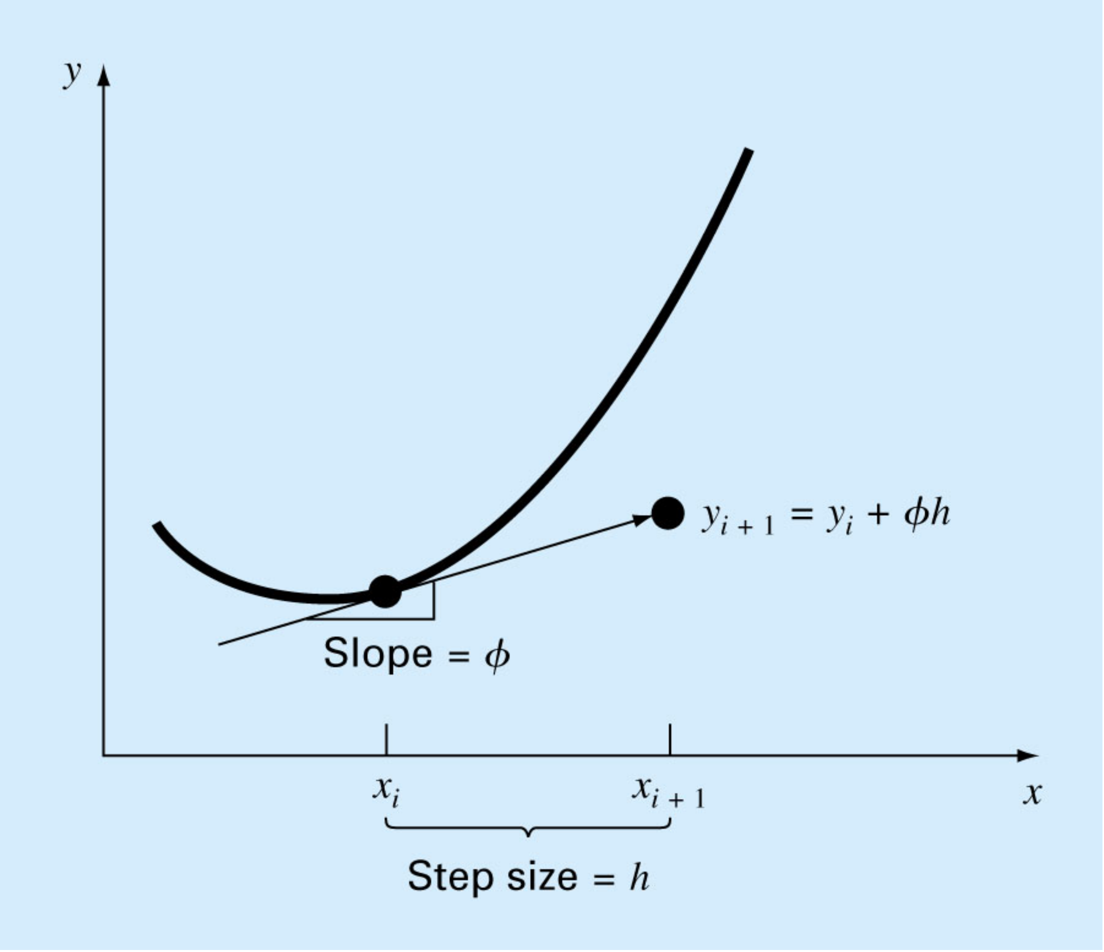
Numerical to solve

New value = old value+ slope$\times$step size

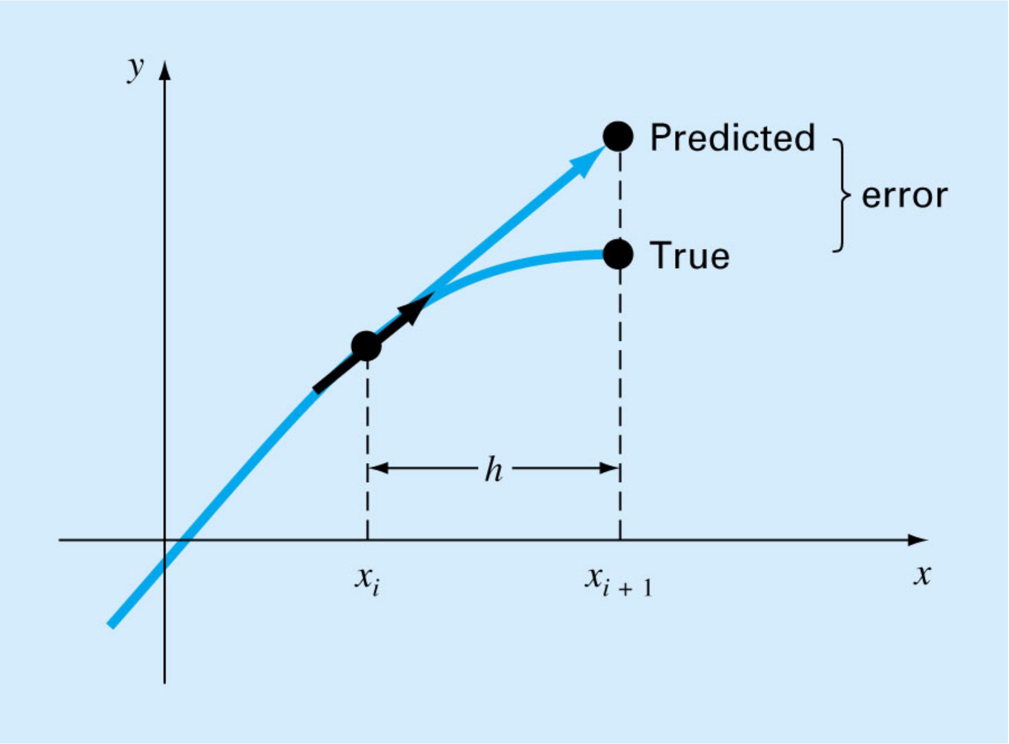
Mathematical term

$$y_{i+1} = y_i + \phi h$$

One-step Methods



Taylor Methods: Euler's method



At x_i $\phi = f\left(x_i, y_i\right)$

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

Example25.1

EXAMPLE 25.1 Euler's Method

Problem Statement. Use Euler's method to numerically integrate Eq. (PT7.13):

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from $x = 0$ to $x = 4$ with a step size of 0.5. The initial condition at $x = 0$ is $y = 1$. Recall that the exact solution is given by Eq. (PT7.16):

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

Solution. Equation (25.2) can be used to implement Euler's method:

$$y(0.5) = y(0) + f(0, 1)0.5$$

where $y(0) = 1$ and the slope estimate at $x = 0$ is

$$f(0, 1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$

Therefore,

$$y(0.5) = 1.0 + 8.5(0.5) = 5.25$$

The true solution at $x = 0.5$ is

$$y = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1 = 3.21875$$

Example25.1

Thus, the error is

$$E_t = \text{true} - \text{approximate} = 3.21875 - 5.25 = -2.03125$$

or, expressed as percent relative error, $\varepsilon_t = -63.1\%$. For the second step,

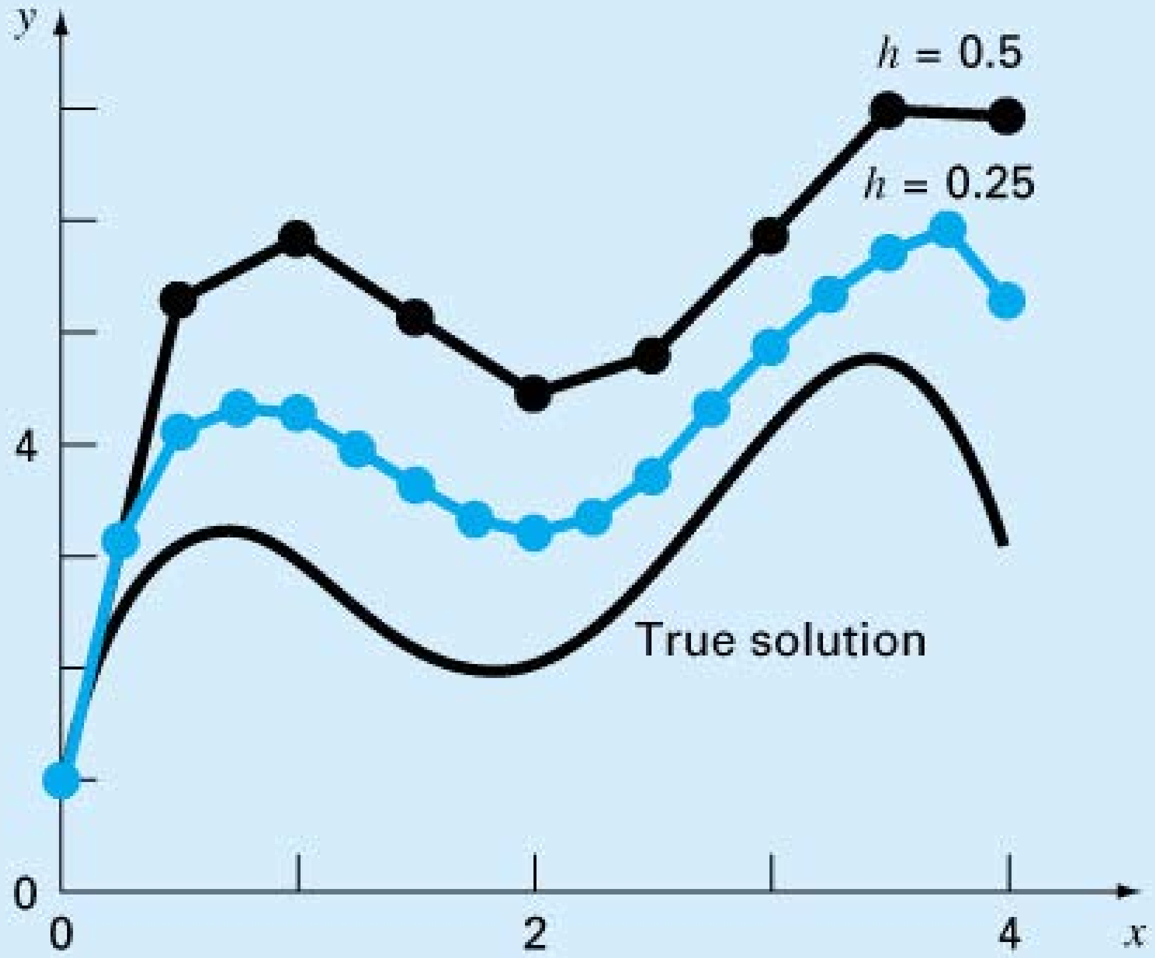
$$y(1) = y(0.5) + f(0.5, 5.25)0.5$$
$$= 5.25 + [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5]0.5$$
$$= 5.875$$

The true solution at $x = 1.0$ is 3.0, and therefore, the percent relative error is −95.8%. The computation is repeated, and the results are compiled in Table 25.1 and Fig. 25.3. Note that,

TABLE 25.1 Comparison of true and approximate values of the integral of $y' = -2x^3 + 12x^2 - 20x + 8.5$, with the initial condition that $y = 1$ at $x = 0$. The approximate values were computed using Euler's method with a step size of 0.5. The local error refers to the error incurred over a single step. It is calculated with a Taylor series expansion as in Example 25.2. The global error is the total discrepancy due to past as well as present steps.

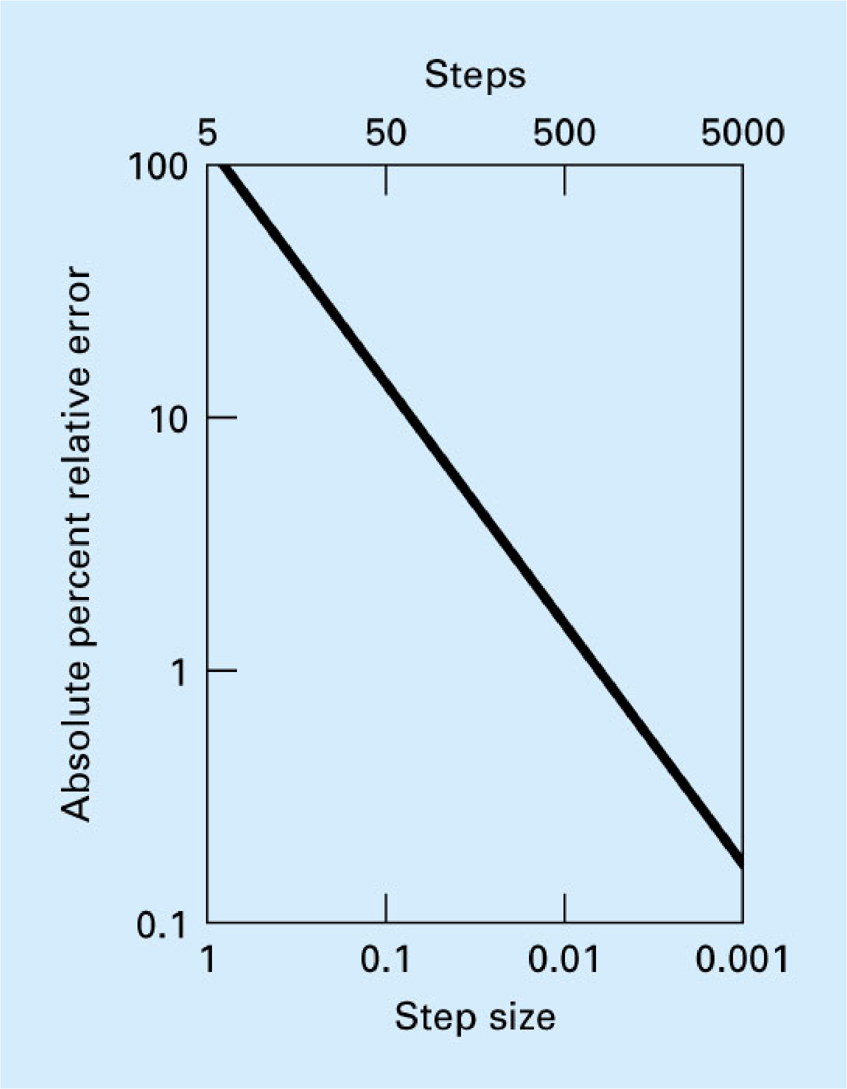
			Percent Relative Error	
x	y_{true}	y_{Euler}	Global	Local
0.0	1.00000	1.00000		
0.5	3.21875	5.25000	−63.1	−63.1
1.0	3.00000	5.87500	−95.8	−28.0
1.5	2.21875	5.12500	131.0	−1.41
2.0	2.00000	4.50000	−125.0	20.5
2.5	2.71875	4.75000	−74.7	17.3
3.0	4.00000	5.87500	46.9	4.0
3.5	4.71875	7.12500	−51.0	−11.3
4.0	3.00000	7.00000	−133.3	−53.0

Example25.1

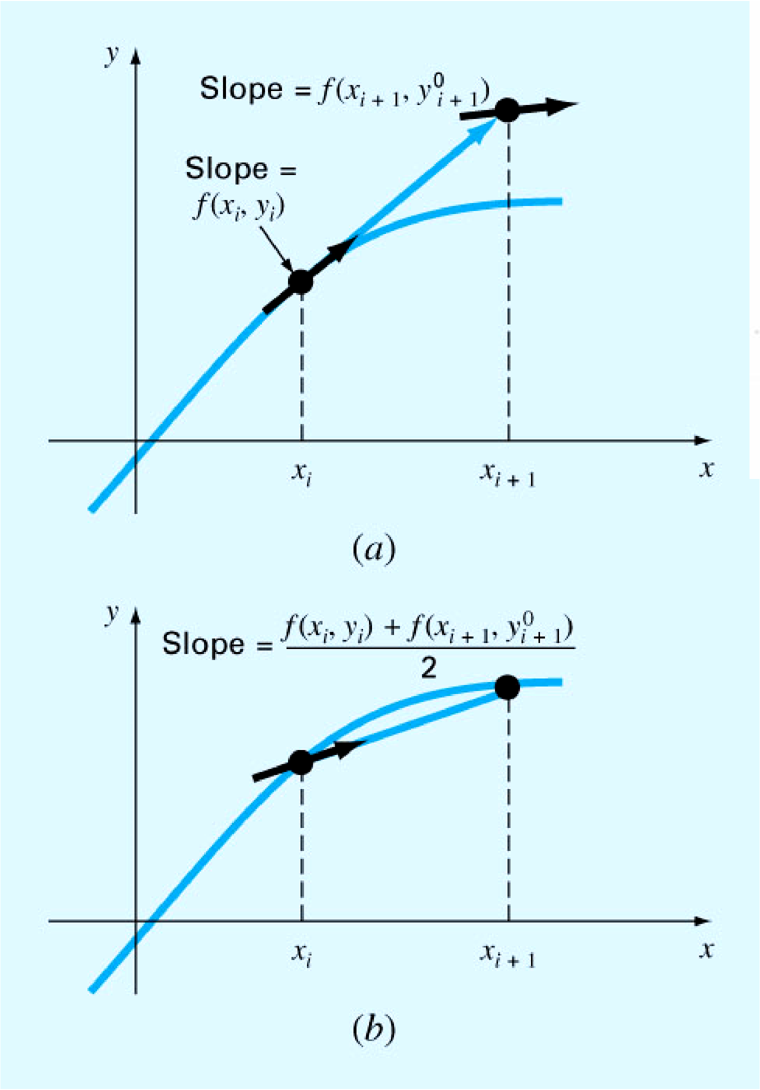


(a)

Effect of step size on the global truncation error



Taylor Methods: Heun's method



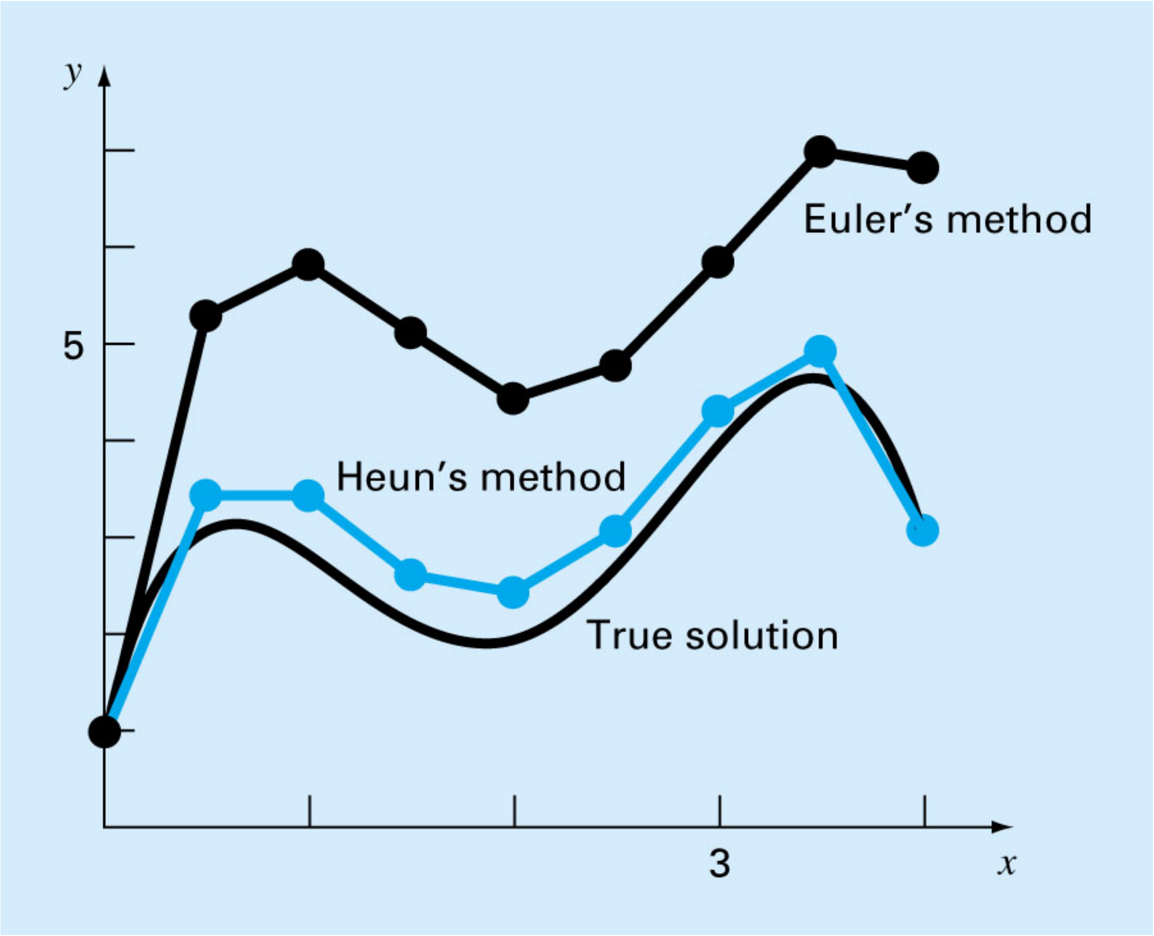
The Heun method is a *predictor-corrector approach*. All the multistep methods to be discussed subsequently in Chap. 26 are of this type. The Heun method is the only one-step predictor-corrector method described in this book. As derived above, it can be expressed concisely as

Predictor (Fig. 25.9*a*): $$y^0_{i+1} = y_i + f(x_i, y_i)h \quad (25.15)$$

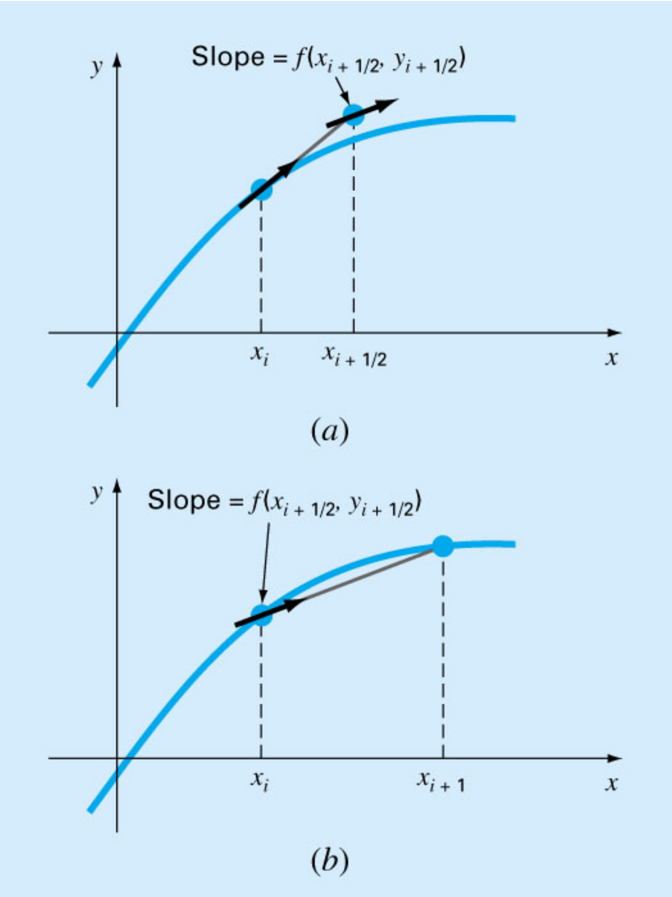
Corrector (Fig. 25.9*b*): $$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2}h \quad (25.16)$$

Note that because Eq. (25.16) has y_{i+1} on both sides of the equal sign, it can be applied in an iterative fashion. That is, an old estimate can be used repeatedly to provide an improved estimate of y_{i+1}. The process is depicted in Fig. 25.10. It should be understood that

Heun's method



Runge-Kutta Methods: Midpoint Method



A value of *y* at the midpoint of the interval

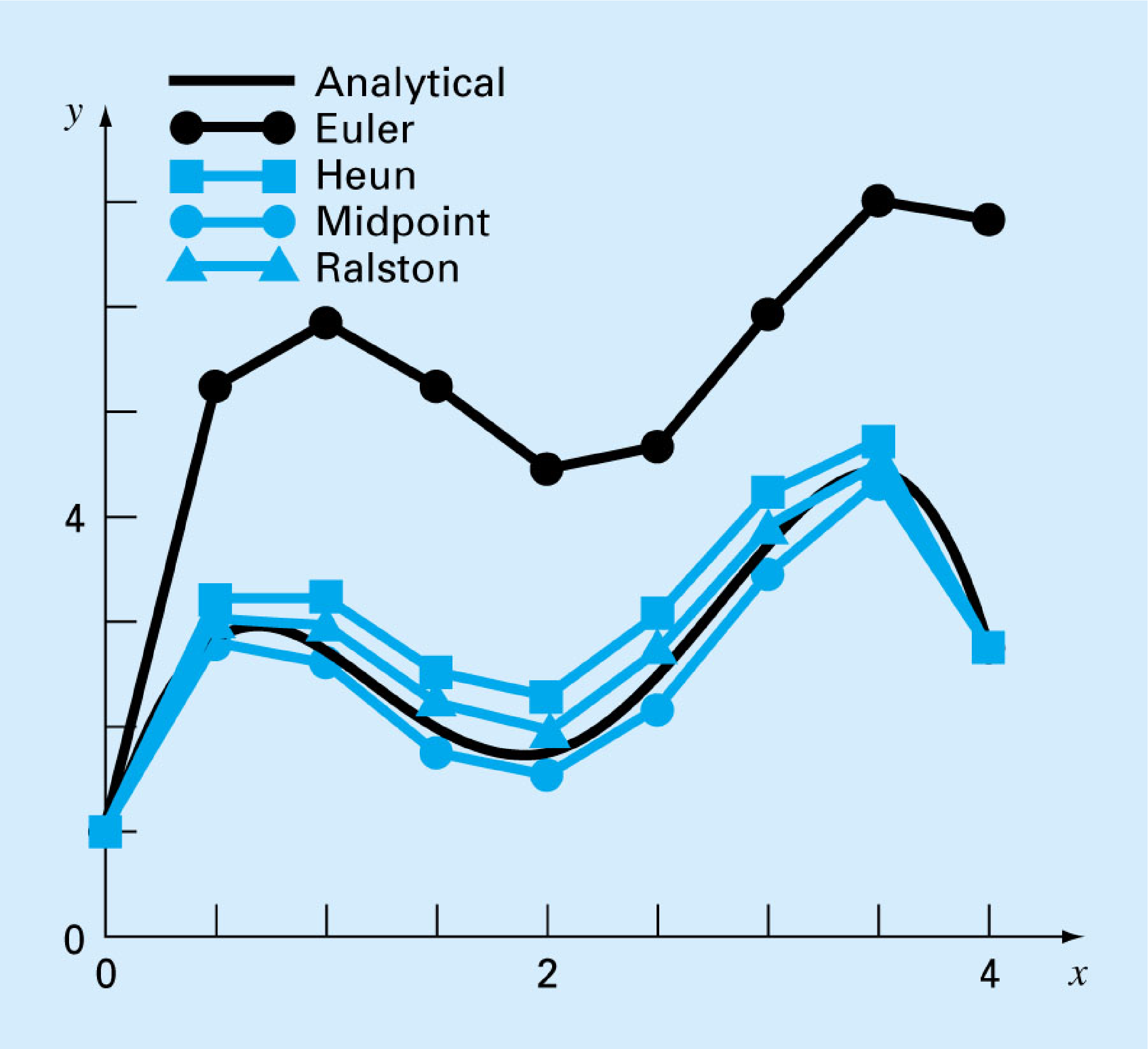
$$y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2}$$

Slope at the midpoint

$$y'_{i+1} = f\left(x_{i+1/2}, y_{i+1/2}\right)$$

The midpoint method

$$y_{i+1} = y_i + f\left(x_{i+1/2}, y_{i+1/2}\right)h$$



Analytical
Euler
Heun
Midpoint
Ralston
y
4
0
0
2
4
x

Example

Problem statement $\frac{dy}{dt} = 3 - 2t + 4\sin 2t$

Initial value y(t=0)=1

Exact solution

$$y = 3t - t^2 - 2\cos 2t + 3$$

Time	True solution
0	1.0000
0.2	1.7179
0.4	2.6466
0.6	3.7153
0.8	4.8184
1.0	5.8323
1.2	6.6348
1.4	7.1244
1.6	7.2366
1.8	6.9535
2.0	6.3073
2.2	5.3747
2.4	4.2650
2.6	3.1030
2.8	2.0089
3.0	1.0797
3.2	0.3736
3.4	-0.0988
3.6	-0.3767
3.8	-0.5425
4.0	-0.7090
4.2	-1.0014
4.4	-1.5378
4.6	-2.4103
4.8	-3.6706